## P.11 HIGHER ORDER DISPERSION IN THE PROPAGATION OF A GRAVITY WAVE PACKET

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To the first order of approximation, the complex amplitude of a wave packet in an anisotropic and dispersive medium is convected with the group of velocity. However, a gravity wave is a vector wave. Its wave packet must be formed by superposition of various wave numbers with corresponding frequencies, as is the case for scalar waves, and additionally by superposing many eigenmodes which also depend on the wave number. To represent the vector wave packet self-consistently, it is found that a gradient term must be included in the expansion. For a Gaussian wave packet, this gradient term is shown to have important implications on the velocity vector as represented by its hodograph. Numerical results show that the hodograph is influenced by the location of the relative position of interest from the center of a Gaussian pulse. Higher order expansion shows that an initial Gaussian wave packet will retain its Gaussian shape as it propagates, but the pulse will spread in all directions with its major axis undergoing a rotation. Numerical results indicate that these higher order dispersive effects may be marginally observable in the atmosphere.

1. Introduction

Of all experiments on atmospheric gravity waves, the wave trains are observed to be, sometimes, quasi-sinusoidal but always to have finite extent in time and in space. To predict the outcome of such waves in the atmosphere requires the study of propagation of a wave packet. By nature, the propagation of such atmospheric waves is both dispersive and anisotropic. Even though the dispersive effects have been studied for some time for both one-dimensional and higher dimensional waves, the additional anisotropic effects do not seem to have received much attention. In this paper both dispersion and anisotropy will be taken into account. It will be shown that when the Earth's rotation is considered, the otherwise closed hodograph may become open for a wave packet. Due to the action of a Coriolis force, a monochromatic wave is accompanied by a hodograph rotating in the clockwise sense in the Northern Hemisphere and counter-clockwise in the Southern Hemisphere. However, the polychromatic nature of a wave packet may change both the ellipse ratio and the sense of rotation. The combined dispersive and anisotropic effects on the propagation of a Gaussian wave packet is investigated in this paper.

2. Mathematical Formulation

For simplicity the atmosphere will be assumed to be inviscid and isothermal. Let the first order perturbations in the atmosphere associated with the propagation of gravity waves be described by a five-dimensional vector $F$ where the first component is proportional to the entropy perturbation per unit mass, the second component is proportional to the pressure perturbation and the third through fifth components are proportional to the perturbed velocity. The last three components of $F$ thus determine the nature of a hodograph.

The linearized hydrodynamic equations can be cast into an equation with a dyadic operator operating on $F$. By a judicious choice of $F$, this operator consists of both time and spatial differentiation but is of constant coefficients. Thus Fourier transform in time and space is permitted and the analysis of the resulting equations produces eigenmodes, $\tilde{u}(\vec{k})$. For a wave packet, the eigenmodes are assumed to have a narrow spectrum. By expanding to the second order, the state vector $F$ can be related to the eigenmodes $\tilde{u}(\vec{k}_0)$ at the carrier wave number $\vec{k}_0$ through the following equation

$$F(\vec{r},t) = [A(\vec{r},t)\tilde{u}(\vec{k}_o) + j \nabla A \bullet \nabla_{k_o} \tilde{u}] \exp j[\omega(\vec{k}_o)t - \vec{k}_o \bullet \vec{r}] \qquad (1)$$

where A is the complex envelope function. Readers interested in the detailed proof should consult Dong and Yeh [1989].

To the first order, the complex envelope function is found to have the form

$$A(\vec{r},t) = A(\vec{R}) \qquad (2)$$

where $\vec{R} = \vec{r} - \vec{v}_g t$. In this case, the envelope is thus convected with a group velocity $\vec{v}_g$ undistorted. Even in this order, the state vector $\vec{F}$ given by (1) in general may have a non-negligible gradient term. When this happens, $\vec{F}$ is not a simple convection alone; it undergoes a time-dependent change while being convected. This is most easily demonstrated from numerical calculations of a Gaussian pulse. The results are shown in Figure 1 where the hodograph at three fixed positions are plotted as a function of time starting at its peak. In order to illustrate the three-dimensional nature of the velocity orbits, four different projections have been used. They show that even at a point where the Gaussian peak would pass through (i.e., at the origin in Figure 1), the hodograph is still clockwise but open. For a point away from the Gaussian peak, the ellipse ratio or the sense of rotation or both may change as demonstrated by plots of Figure 1 at plus or minus five wavelengths away from the Gaussian peak.

When expanding to the second order, the complex envelope function A itself is found to distort with time as it propagates with the group velocity, since now, instead of (2) it becomes

$$A(\vec{r},t) = A(\vec{R},t) \qquad (3)$$

One way to describe the variation of A as it propagates is to define a surface S on which the magnitude is $\exp(-1/2)$ of its value at the Gaussian peak, i.e.,

$$| A(\vec{R},t)| = | A(\vec{o},t)| \exp(-1/2) \qquad (4)$$

It has been found that the surface S is a surface of spheroid, the three axes of which $L_1$, $L_2$ and $L_3$, will change with time. The peak of the Gaussian wave packet $| A(\vec{o},t)|$ normalized by its initial value $A(\vec{o},o)| = A_0$ as given by the ratio $A/A_0$ in Figure 2 decreases with time. This is caused entirely by dispersion. Furthermore, this ellipsoid undergoes a rotation as indicated by the angle $\alpha$. Both the pulse dispersion and the rotation of the ellipsoid are clearly depicted in Figure 3 which plots the projection of the S surface on the xz-plane at successive instants. Successively, the initially sharply peaked Gaussian pulse becomes reduced at its peak value, coupled simultaneously with a more dispersed pulse in the spatial domain and a rotation of its major axis.

3. Conclusion

A theory describing the propagation of a wave packet in a dispersive and anisotropic medium has been developed. This theory is applied to gravity waves. It is found that important new features are manifested. The calculations that have been carried out assume a Gaussian pulse with half-width equal to five wavelengths. As many observations, even in the best of times, only show a few oscillations, there is need to extend the study to narrower Gaussian pulses for which the dispersive effects are expected to be even more important. In preparation for such investigations, the theory must be extended to a higher order than that given by (1), which is now in progress.

Reference

Dong, B., and K. C. Yeh, Polarization and dispersion of a Gaussian gravity-wave packet, *Ann. Geophysicae*, to appear in January 1989 issue.
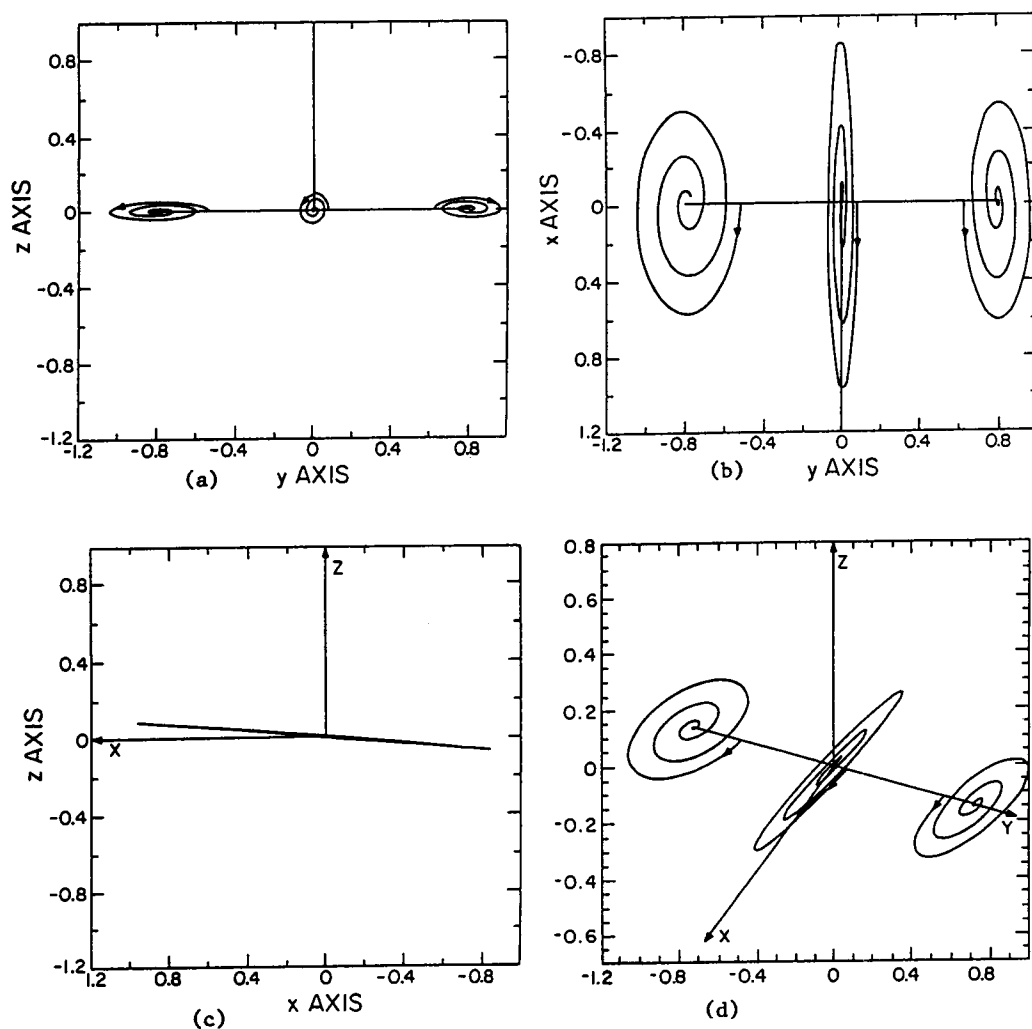


**Figure 1.** Projection of 3-D hodographs of the velocities of a Gaussian gravity wave packet having a period 55.6 min and a propagation vector 5° away from downward direction. (a) front view (from x-axis), (b) top view (from z-axis), (c) side view (from y-axis), and (d) view from a point at $\theta = 25°$ and $\phi = 65°$.
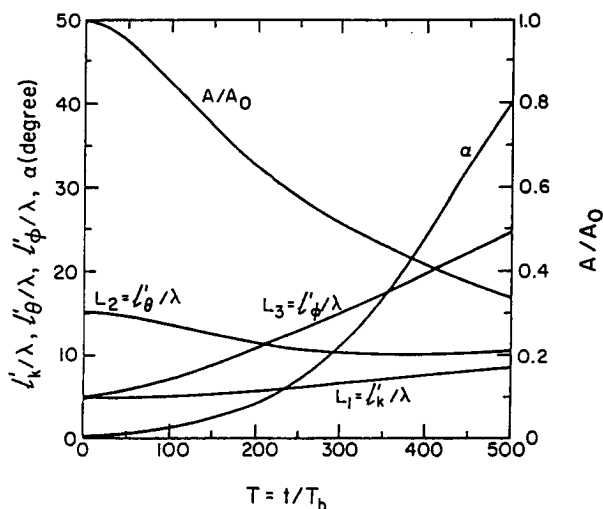
Figure 2. The propagation of a Gaussian packet can be described by the motion of a spheroid. The three axes of the ellipsoid $L_1$, $L_2$, $L_3$ normalized by the wavelength $\lambda$ vary with time as shown. The temporal variation of the magnitude at the center of the Gaussian packet normalized by its initial value at $t = 0$ is shown as the curve $A/A_0$ using the right scale. The rotating angle $\alpha$ uses the left scale. The initial parameters of the wave packet are: $1_k = 5\lambda$, $1_\theta = 15\lambda$, $1_\phi = 5\lambda$, $\theta_0 = 175°$, $\phi_0 = 0°$.
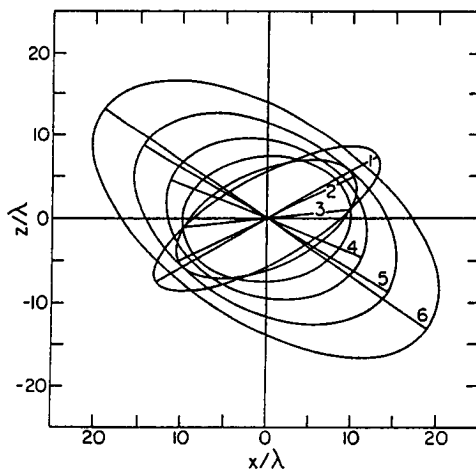


Figure 3. The projection of the ellipsoidal surface S at successive times on the plane xoz. The time increment used is 200 $T_b$. As time increases, the spreading and rotation of the Gaussian packet can be visualized. The angle $\alpha$ determines the orientation of the major axis of the ellipse and is measured from the axis at $t = 0$ marked by 1. The initial parameters of the wave packet are identical to those used in Figure 3.